

Balanced Allocation: Patience is not a Virtue

John Augustine*

William K. Moses Jr.[†]Amanda Redlich[‡]Eli Upfal[§]

Abstract

Load balancing is a well-studied problem, with balls-in-bins being the primary framework. The greedy algorithm **Greedy** $[d]$ of Azar et al. places each ball by probing $d > 1$ random bins and placing the ball in the least loaded of them. It ensures a maximum load that is exponentially better than the strategy of placing each ball uniformly at random. Vöcking showed that a slightly asymmetric variant, **Left** $[d]$, provides a further significant improvement. However, this improvement comes at an additional computational cost of imposing structure on the bins.

Here, we present a fully decentralized and easy-to-implement algorithm called **FirstDiff** $[d]$ that combines the simplicity of **Greedy** $[d]$ and the improved balance of **Left** $[d]$. The key idea in **FirstDiff** $[d]$ is to probe until a different bin size from the first observation is located, then place the ball. Although the number of probes could be quite large for some of the balls, we show that **FirstDiff** $[d]$ requires only d probes on average per ball (in both the standard and the heavily-loaded settings). Thus the number of probes is no greater than either that of **Greedy** $[d]$ or **Left** $[d]$. More importantly, we show that **FirstDiff** $[d]$ closely matches the improved maximum load ensured by **Left** $[d]$ in both the standard and heavily-loaded settings. We further provide a tight lower bound on the maximum load up to $O(\log \log \log n)$ terms. We additionally give experimental data that **FirstDiff** $[d]$ is indeed as good as **Left** $[d]$, if not better, in practice.

1 Introduction

Load balancing is the study of distributing loads across multiple entities such that the load is minimized across all the entities. This problem arises naturally in many settings, including the distribution of requests across multiple servers, in peer-to-peer networks when requests need to be spread out amongst the participating nodes, and in hashing. Much research has focused on practical implementations of solutions to these problems [10, 5, 11].

Our work builds on several classic algorithms in the theoretical balls-in-bins model in which m balls are to be placed sequentially into n bins and each ball probes the load in random bins in order to make its choice. Here we give a new algorithm, **FirstDiff**, which performs as well as the best known algorithm while being significantly easier to implement.

The allocation time for a ball is the number of probes made to different bins before we place the ball into a bin. The challenge is to balance the allocation time versus the maximum bin load. For example, using one probe per ball, i.e. placing each ball uniformly at random, the maximum load of any bin when $m = n$ will be $\frac{\log n}{\log \log n}(1 + o(1))$ (with high probability¹) and total allocation time of $O(n)$ [9]. On the other hand, using d probes per ball and placing the ball in the lightest bin, i.e. **Greedy** $[d]$, first studied by Azar et al [1], decreases the maximum load to $\frac{\log \log n}{\log d} + O(1)$ with allocation time of $O(nd)$. In other words, using $d \geq 2$ choices improves the maximum load exponentially, at a linear allocation cost.

Quite surprisingly, Vöcking [13] introduced a slightly asymmetric algorithm, **Left** $[d]$, which guaranteed (w.h.p.) a maximum load of $\frac{\log \log n}{d \log \phi_d} + O(1)$ (where ϕ_d is a constant between 1 and 2) using the same allocation

*Department of Computer Science & Engineering, Indian Institute of Technology Madras, Chennai, India. augustine@cse.iitm.ac.in. Supported by the IIT Madras New Faculty Seed Grant, the IIT Madras Exploratory Research Project, and the Indo-German Max Planck Center for Computer Science (IMPECS).

[†]Department of Computer Science & Engineering, Indian Institute of Technology Madras, Chennai, India. wkmjr@cse.iitm.ac.in

[‡]Department of Mathematics, Bowdoin College, ME, USA. aredlich@bowdoin.edu

[§]Department of Computer Science, Brown University, RI, USA. eli@cs.brown.edu

¹We use the phrase “with high probability” (or w.h.p. in short) to denote probability of the form $1 - n^{-c}$ for some suitable $c > 0$. Furthermore, every log in this paper is to base 2 unless otherwise mentioned.

time when $m = n$. This was extended to the heavily-loaded case by Berenbrink et al. [2]. However, $\text{Left}[d]$ in [13] and [2] relies on additional processing. Bins are initially sorted into groups and treated differently according to group membership. Thus practical implementation, especially in distributed settings, requires significant computational effort beyond the probes.

Our Contribution. We present a new algorithm, $\text{FirstDiff}[d]$. This algorithm requires no pre-sorting of bins; instead $\text{FirstDiff}[d]$ uses real-time feedback to adjust its number of probes for each ball².

The natural comparison is with the classic $\text{Greedy}[d]$ algorithm; $\text{FirstDiff}[d]$ uses the same number of probes, on average, as $\text{Greedy}[d]$ but produces a significantly smaller maximum load. In fact, the maximum load is as small as that of $\text{Left}[d]$ when $m = n$, and comparable to $\text{Left}[d]$ when heavily loaded, but with much lower computational overhead.

This simpler implementation makes $\text{FirstDiff}[d]$ especially suitable for practical applications; it is amenable to parallelization, for example, and requires no central control or underlying structure. Some applications have a target maximum load and minimize the necessary number of probes. From this perspective, our algorithm again improves on $\text{Greedy}[d]$: the maximum load of $\text{FirstDiff}[\log d]$ is comparable to $\text{Greedy}[d]$, and uses exponentially fewer probes per ball.

Theorem 1. *Use $\text{FirstDiff}[d]$ to allocate n balls into n bins. The average number of probes required per ball is d on expectation. Furthermore, the maximum load of any bin is at most $\frac{\log \log n}{\Theta(d)} + O(1)$ with high probability when $n \geq \max(2, n_0)$, where n_0 is the smallest value of n such that for all $n > n_0$, $36 \log n \left(\frac{72e \log n}{5n} \right)^4 \leq \frac{1}{n^2}$.*

Theorem 2. *Use $\text{FirstDiff}[d]$ to allocate m balls into n bins. When $m \geq c_1 \frac{2^{\Theta(d)}}{\Theta(d)} n \log n$, where c_1 is an appropriately chosen constant, it takes d probes on average to place every ball on expectation. Furthermore, for an absolute constant c ,*

$$\Pr \left(\text{Max. load of any bin} > \frac{m}{n} + \frac{\log \log n}{\Theta(d)} + c \log \log \log n \right) \leq c(\log \log n)^{-4}.$$

Our technique for proving that the average number of probes is bounded is novel to the best of our knowledge. As the number of probes required by each ball is dependent on the configuration of the balls-in-bins at the time the ball is placed, the naive approach to computing its expected value quickly becomes too conditional. Instead, we show that this conditioning can be eliminated by careful application of the law of total probability, leading to a proof that is then quite simple. The heavily-loaded case is significantly more complex than the $m = n$ case; however the basic ideas remain the same.

The upper bound on the maximum load is proved using the layered induction technique. However, because $\text{FirstDiff}[d]$ is a dynamic algorithm, the standard recursion used in layered induction must be altered. We use coupling and some more complex analysis to adjust the standard layered induction to this context.

Subsequently, we provide a tight lower bound on the maximum load for a broad class of algorithms which use variable probing.

Theorem 4. *Let $\text{Alg}[k]$ be any algorithm that places m balls into n bins, where $m \geq n$, sequentially one by one and satisfies the following conditions:*

1. *At most k probes are used to place each ball.*
2. *For each ball, each probe is made uniformly at random to one of the n bins.*
3. *For each ball, each probe is independent of every other probe.*

The maximum load of any bin after placing all m balls using $\text{Alg}[k]$ is at least $\frac{m}{n} + \frac{\ln \ln n}{\ln k} - \Theta(1)$ with high probability.

We use the above theorem to provide a lower bound on the maximum load of $\text{FirstDiff}[d]$, which is tight upto $O(\log \log \log n)$ terms.

²Thus we are concerned with the *average* number of probes per ball throughout this paper.

Theorem 5. *The maximum load of any bin after placing m balls into n bins using FirstDiff[d] is at least $\frac{m}{n} + \frac{\ln \ln n}{\Theta(d)} - \Theta(1)$ with high probability.*

Related Work. Several other algorithms in which the number of probes performed by each ball is adaptive in nature have emerged in the past. Czumaj and Stemann [4] present several “threshold” algorithms. In one family of algorithms, each load value has an associated threshold and a ball is placed when the number of probes that were made to find a bin of a particular load exceeded the associated threshold. In another family of algorithms, each ball probes bins until it finds one whose load is within some predetermined threshold. Although the authors provide a number of bounds on maximum load and the allocation time, many of their algorithms don’t extend naturally to cases where the number of balls is more than the number of bins. Furthermore, computing their threshold values often depends on the knowledge (typically unavailable in practical applications) of the total number of balls and bins.

More recently, Berenbrink et al. [3] give two adaptive threshold based algorithms, where the threshold depends either on the total number of bins or the total number of balls placed thus far. This depends on a different sort of global knowledge that again can be impractical. Our algorithm is unique in that it requires no memory at all; it is able to make decisions based on the probed bins’ load values alone.

Definitions. In the course of this paper we will use several terms from probability theory, which we define below for convenience.

Consider two Markov chains A_t and B_t over time $t \geq 0$ with state spaces S_1 and S_2 respectively. A *coupling* (cf. [6]) of A_t and B_t is a Markov chain (A_t, B_t) over time $t \geq 0$ with state space $S_1 \times S_2$ such that A_t and B_t maintain their original transition probabilities.

Consider two vectors $u, v \in \mathbb{Z}^n$. Let u' and v' be permutations of u and v respectively such that $u'_i \geq u'_{i+1}$ and $v'_i \geq v'_{i+1}$ for all $1 \leq i \leq n-1$. We say u *majorizes* v (or v *is majorized by* u) when

$$\sum_{j=1}^i u'_j \geq \sum_{j=1}^i v'_j, \forall 1 \leq i \leq n.$$

For a given allocation algorithm C which places balls into n bins, we define the *load vector* $u^t \in (\mathbb{Z}^*)^n$ of that process after t balls have been placed as follows: the i^{th} index of u^t denotes the load of the i^{th} bin (we can assume a total order on the bins according to their IDs). Note that $u^t, t \geq 0$, is a Markov chain.

Consider two allocation algorithms C and D that allocate m balls. Let the load vectors for C and D after t balls have been placed using the respective algorithms be u^t and v^t respectively. We say that C *majorizes* D (or D *is majorized by* C) if there is a coupling between C and D such that u_t majorizes v_t for all $0 \leq t \leq m$.

Berenbrink et al. [2] provide an illustration of the above ideas being applied in the load balancing context.

Organization of Paper. The structure of this paper is as follows. In Section 2, we define the model formally and present the FirstDiff[d] algorithm. We then analyse the algorithm when $m = n$ in Section 3 and give a proof that the expected total number of probes used by FirstDiff[d] to place n balls is nd , while the maximum bin load is still upper bounded by $\frac{\log \log n}{\Theta(d)} + O(1)$ with high probability. We provide the analysis of the algorithm when $m > n$ in Section 4, namely that the expected number of probes is on average d per ball and the maximum bin load is upper bounded by $\frac{m}{n} + \frac{\log \log n}{\Theta(d)} + O(\log \log \log n)$ with probability close to 1. We provide a matching lower bound for maximum bin load tight up to the $O(\log \log \log n)$ term for algorithms with variable number of probes and FirstDiff[d] in particular in Section 5. In Section 6, we show experimentally that our FirstDiff[d] algorithm indeed results in a maximum load that is comparable to Left[d] when $m = n$. Finally, we provide some concluding remarks and scope for future work in Section 7.

2 The FirstDiff[d] Algorithm

The idea behind this algorithm is to use probes more efficiently. In the standard d -choice model, effort is wasted in some phases. For example, early on in the distribution, most bins have size 0 and there is no need to search before placing a ball. On the other hand, more effort in other phases would lead to significant improvement. For example, if $.9n$ balls have been distributed, most bins already have size at least 1 and thus it is harder to avoid creating a bin of size 2. FirstDiff[d] takes this variation into account by probing until it finds a difference, then makes its decision.

This algorithm uses probes more efficiently than other, fixed-choice algorithms, while still having a balanced outcome. Each ball probes at most $2^{\Theta(d)}$ bins (where $d \geq 2$ and by extension $2^{\Theta(d)} \geq 2$) uniformly at random until it has found two bins with different loads (or a bin with zero load) and places the ball in the least loaded of the probed bins (or the zero loaded bin). If all $2^{\Theta(d)}$ probed bins are equally loaded, the ball is placed (without loss of generality) in the last probed bin. The pseudocode for FirstDiff[d] is below.

Algorithm 1 FirstDiff[d]

(Assume $d \geq 2$. The following algorithm is executed for each ball.)

- 1: Repeat $2^{\Theta(d)}$ times
 - 2: Probe a new bin chosen uniformly at random
 - 3: **if** The probed bin has zero load **then**
 - 4: Place the ball in the probed bin and exit
 - 5: **if** The probed bin has load that is different from those probed before **then**
 - 6: Place the ball in the least loaded bin (breaking ties arbitrarily) and exit
 - 7: Place the ball in the last probed bin
-

As we can see, the the manner in which a ball can be placed using FirstDiff[d] can be classified as follows:

1. All probes were made to bins of the same load.
2. One or more probes were made to bins of larger load followed by a probe to a bin of lesser load.
3. One or more probes were made to bins of lesser load followed by a probe to a bin of larger load.

3 Analysis of FirstDiff[d] when $m = n$

Theorem 1. *Use FirstDiff[d] to allocate n balls into n bins. The average number of probes required per ball is d on expectation. Furthermore, the maximum load of any bin is at most $\frac{\log \log n}{\Theta(d)} + O(1)$ with high probability when $n \geq \max(2, n_0)$, where n_0 is the smallest value of n such that for all $n > n_0$, $36 \log n \left(\frac{72e \log n}{5n} \right)^4 \leq \frac{1}{n^2}$.*

Proof. First, we show that the average number of probes per ball is d . Subsequently, we show that the maximum load at the end of placing all n balls is as desired.

3.1 Proof of Number of Probes

Lemma 1. *The expected number of probes required to place $m = n$ balls into n bins using FirstDiff[d] is nd .*

Proof. Let k be the maximum number of probes allowed to be used by FirstDiff[d] per ball, i.e. $k = 2^{\Theta(d)}$. Let the balls be indexed from 1 to n in the order in which they are placed. Our proof proceeds in two phases. For a value of T that will be fixed subsequently, the first $T + 1$ balls are analysed in the first phase and remaining balls are analysed in the second. Consider the ball indexed by t , $1 \leq t \leq n$.

Phase One: $t \leq T + 1$. Notice that the expected number of probes necessary to place one ball for an arbitrary configuration of t balls is upper bounded by the expected number of probes necessary for t bins of load 1 and $n - t$ bins of load 0. You can see this by a simple sequence of couplings.

First, choose some arbitrary configuration with $n\alpha_i$ bins of size i for $i = 0, 1, 2, \dots$. That configuration will be probed until bins of two different sizes are discovered, i.e. until the set probed intersects two distinct α_i and α_j . Couple this with the configuration that has $\sum_{i=1}^n n\alpha_i$ bins of size 1 and the rest of size 0. This configuration requires more probes than the original configuration; it continues until the set probed intersects α_0 and $\alpha_{\neq 0}$. Second, note that the configuration with t bins of size 1 and $n - t$ bins of size 0 requires even more probes than this one (on expectation).

So it is enough to find the expected number of probes for t bins of load 1 and $n - t$ bins of load 0. The expected number of probes used is upper bounded by the expected number of probes until a size-0 bin appears. This is of course $n/(n - t)$. The overall expected number of probes for the first $T + 1$ steps is thus upper bounded by

$$\begin{aligned} \sum_{t=0}^T \frac{n}{n-t} &= n \sum_{i=n-T}^n \frac{1}{i} \sim n(\log n - \log(n-T)) \\ &= n \log \left(\frac{n}{n-T} \right). \end{aligned}$$

Phase Two: $t > T + 1$. Rather than analysing in detail, we use the fact that the number of probes at each stage is at most k . So the number of probes overall in this phase is at most $k(n - T - 1)$.

Now we will find T such that the number of probes in each phase is bounded by $n \log k$, to show that the overall number of probes is $O(n \log k)$. In Phase One the number of probes is

$$n \log \left(\frac{n}{n-T} \right) \leq n \log k.$$

Solve to get $T = n(1 - 1/k)$. In Phase Two, the total number of probes is

$$k(n - T - 1) = k(n - n(1 - 1/k) - 1) = n - k.$$

So the sum is indeed $(1 + o(1))n \log k$. When $k = 2^{\Theta(d)}$ for an appropriately chosen constant, the expected number of probes to place all n balls is nd probes, as desired. \square

3.2 Proof of Maximum Load

Lemma 2. *The maximum load in any bin after using FirstDiff[d] to allocate n balls into n bins is at most $\frac{\log \log n}{\Theta(d)} + O(1)$ with high probability when $n \geq \max(2, n_0)$, where n_0 is the smallest value of n such that for all $n > n_0$, $36 \log n \left(\frac{72e \log n}{5n} \right)^4 \leq \frac{1}{n^2}$.*

Proof. While the proof follows along the lines of the standard layered induction argument [6, 12], we have to make a few non-trivial adaptations to fit our context where the number of probes is not fixed.

Let k be the maximum number of probes allowed to be used by FirstDiff[d] per ball, i.e. $k = 2^{\Theta(d)}$. Define v_i as the fraction of bins of load at least i after n balls are placed. Define u_i as the number of balls of height at least i after n balls are placed. It is clear that $v_i * n \leq u_i$.

We wish to show that the $\mathbf{Pr}(\text{Max. load} \geq \frac{\log \log n}{\log k} + \gamma) \leq \frac{1}{n^c}$ for some constants $\gamma \geq 1$ and $c \geq 1$. Set $i^* = \frac{\log \log n}{\log k} + 11$ and $\gamma = 15$. Equivalently, we wish to show that $\mathbf{Pr}(v_{i^*+4} > 0) \leq \frac{1}{n^c}$ for some constant $c \geq 1$.

In order to aid us in this proof, let us define a non-increasing series of numbers $\beta_{11}, \beta_{12}, \dots, \beta_{i^*}$ as upper bounds on $v_{11}, v_{12}, \dots, v_{i^*}$. Let us set $\beta_{11} = \frac{1}{11}$.

Now,

$$\begin{aligned}
\Pr(v_{i^*+4} > 0) &= \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \Pr(v_{i^*} \leq \beta_{i^*}) \\
&\quad + \Pr(v_{i^*+4} > 0 | v_{i^*} > \beta_{i^*}) \Pr(v_{i^*} > \beta_{i^*}) \\
&\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) + \Pr(v_{i^*} > \beta_{i^*}) \\
&\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \\
&\quad + \Pr(v_{i^*} > \beta_{i^*} | v_{i^*-1} \leq \beta_{i^*-1}) \Pr(v_{i^*-1} \leq \beta_{i^*-1}) \\
&\quad + \Pr(v_{i^*} > \beta_{i^*} | v_{i^*-1} > \beta_{i^*-1}) \Pr(v_{i^*-1} > \beta_{i^*-1}) \\
&\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \\
&\quad + \sum_{i=12}^{i^*} \Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}) + \Pr(v_{11} > \beta_{11})
\end{aligned} \tag{1}$$

Here, $\Pr(v_{11} > \beta_{11}) = 0$. We now find upper bounds for the remaining two terms in the above equation.

We now derive a recursive relationship between the β_i 's for $i \geq 11$. β_{i+1} acts as an upper bound for the fraction of bins of height at least $i+1$ after n balls are placed. In order for a ball placed to land up at height at least $i+1$, one of 3 conditions must occur:

- All k probes are made to bins of height at least i .
- Several probes are made to bins of height at least i and one is made to a bin of height at least $i+1$.
- One probe is made to a bin of height at least i and several probes are made to bins of height at least $i+1$.

Thus the probability that a ball is placed at height at least $i+1$, conditioning on $v_j \leq \beta_j$ for $j \leq i+1$ at that time, is

$$\begin{aligned}
&\leq \beta_i^k + \beta_i \beta_{i+1} (1 + \beta_i + \beta_i^2 + \dots + \beta_i^{k-2}) \\
&\quad + \beta_i \beta_{i+1} (1 + \beta_{i+1} + \beta_{i+1}^2 + \dots + \beta_{i+1}^{k-2}) \\
&\leq \beta_i^k + \beta_i \beta_{i+1} \left(\frac{1 - \beta_i^{k-1}}{1 - \beta_i} + \frac{1 - \beta_{i+1}^{k-1}}{1 - \beta_{i+1}} \right) \\
&\leq \beta_i^k + \beta_{11} \beta_{i+1} \left(2 * \frac{1}{1 - \beta_{11}} \right) \\
&\leq \beta_i^k + \frac{2\beta_{i+1}}{10}
\end{aligned}$$

Let $v_{i+1}(t)$ be the fraction of bins with load at least $i+1$ after the $1 \leq t \leq n$ ball is placed in a bin.

Let $t^* = \min[\arg \min_t v_{i+1}(t) > \beta_{i+1}, n]$, i.e. t^* is the first t such that $v_{i+1}(t) > \beta_{i+1}$ or n if there is no such t . The probability that $t^* < n$ is bounded by the probability that a binomial random variable $B(n, \leq \beta_i^k + \frac{2\beta_{i+1}}{10})$ is greater than $\beta_{i+1}n$.

Fix $\beta_{i+1} = \frac{10}{3} \beta_i^k \geq \frac{2n(\beta_i^k + \frac{2\beta_{i+1}}{10})}{n}$. Then using a Chernoff bound, we can say that with high probability, $t^* = n$ or $v_{i+1} \leq \beta_{i+1}$, so long as $e^{-\frac{(\beta_i^k + \frac{2\beta_{i+1}}{10})}{3}} = O(\frac{1}{n^c})$ for some constant $c \geq 1$.

Now, so long as $\beta_{i+1} \geq \frac{18 \log n}{n}$, $e^{-\frac{(\beta_i^k + \frac{2\beta_{i+1}}{10})}{3}} = O(\frac{1}{n^c})$. Notice that at $i = i^*$, the value of β_i dips below $\frac{18 \log n}{n}$. This can be seen by solving the recurrence with $\log \beta_{11} = -\log 11$ and $\log \beta_{i+1} = \log(\frac{10}{3}) + k \log \beta_i$.

$$\begin{aligned}
\log \beta_{i^*} &= \log \left(\frac{10}{3} \right) (1 + k + k^2 + \dots + k^{\log_k \log n - 1}) \\
&\quad + k^{\log_k \log n} (-\log 11) \\
&= \log \left(\frac{10}{3} \right) \left(\frac{k^{\log_k \log n} - 1}{k - 1} \right) - (\log n)(\log 11) \\
&\leq (\log n) \left(\log \left(\frac{10}{3} \right) - \log 11 \right) \\
&\leq -1.7 \log n
\end{aligned}$$

Therefore, as it is, $\beta_{i^*} \leq \frac{1}{n^{1.7}} \leq \frac{18 \log n}{n}$ when $n \geq 2$. In order to keep the value of β_i at least at $\frac{18 \log n}{n}$, we set

$$\beta_{i+1} = \max \left(\frac{10}{3} \beta_i^k, \frac{18 \log n}{n} \right) \quad (2)$$

With the values of β_i defined, we proceed to bound $\Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1})$, $\forall 12 \leq i \leq i^*$. For a given i ,

$$\begin{aligned}
\Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}) &= \Pr(nv_i > n\beta_i | v_{i-1} \leq \beta_{i-1}) \\
&\leq \Pr(u_i > n\beta_i | v_{i-1} \leq \beta_{i-1})
\end{aligned}$$

We upper bound the above inequality using the following idea. Let Y_r be an indicator variable set to 1 when the following 2 conditions are met: (i) the r^{th} ball placed is of height at least i and (ii) $v_{i-1} \leq \beta_{i-1}$. Y_r is set to 0 otherwise. Now for all $1 \leq r \leq n$, the probability that $Y_r = 1$ is upper bounded by $\beta_{i-1}^k + \frac{2}{10}\beta_i \leq \frac{3}{10}\beta_i + \frac{2}{10}\beta_i \leq \frac{\beta_i}{2}$. Therefore, the probability that the number of balls of height at least i exceeds β_i is upper bounded by $\Pr(B(n, \frac{\beta_i}{2}) > n\beta_i)$, where $B(\cdot, \cdot)$ is a Binomial random variable with given parameters.

According to Chernoff's bound, for $0 < \delta \leq 1$, $\Pr(X \geq (1 + \delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$, where X is the sum of independent Poisson trials and μ is the expectation of X . If we set $\delta = 1$, then we have

$$\begin{aligned}
\Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}) &\leq \Pr(B(n, \frac{\beta_i}{2}) > n\beta_i) \\
&\leq e^{-\frac{n \cdot (\frac{\beta_i}{2})}{3}} \\
&\leq e^{-\frac{n \cdot (\frac{18 \log n}{n})}{6}} \quad (\text{since } \beta_i \geq \frac{18 \log n}{n}, \forall i \leq i^*) \\
&\leq \frac{1}{n^3}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sum_{j=12}^{i^*} \Pr(v_j > \beta_j | v_{j-1} \leq \beta_{j-1}) \leq \frac{\log \log n}{n^3} \\
\implies \sum_{j=l+1}^{i^*} \Pr(v_j > \beta_j | v_{j-1} \leq \beta_{j-1}) &\leq \frac{1}{2n^2} \quad (\text{since } n \geq 2)
\end{aligned} \quad (3)$$

Finally, we need to upper bound

$\Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*})$. Consider a particular bin of load at least i^* . Now the probability that a ball will fall into that bin is

$$\begin{aligned}
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + 2\beta_{i^*+1} \left(\frac{1}{1 - \beta_{i^*+1}^k} \right) \right) \\
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + 2\beta_{i^*+1} \left(\frac{1}{1 - \beta_{11}} \right) \right) \\
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + \frac{22}{10} \beta_{i^*} \right) \text{ (since } \beta_i \text{ is a non-increasing function)} \\
&\leq \frac{1}{n} \cdot \frac{32}{10} \cdot \beta_{i^*} \text{ (since } k \geq 2 \text{ and } \beta_{i^*} \leq 1)
\end{aligned}$$

Now, we upper bound the probability that 4 balls fall into a given bin of load at least i^* and then use a union bound over all the bins of load at least i^* to show that the probability that the fraction of bins of load at least β_{i^*+4} exceeds 0 is negligible.

First, the probability that 4 balls fall into a given bin of load at least β_{i^*} is

$$\begin{aligned}
&\leq \Pr(B(n, (\frac{1}{n} \cdot \frac{32}{10} * \beta_{i^*})) \geq 4) \\
&\leq \binom{n}{4} \left(\frac{1}{n} \cdot \frac{32}{10} \cdot \beta_{i^*} \right)^4 \\
&\leq \left(e \cdot n \cdot \left(\frac{1}{n} \cdot \frac{32}{10} \cdot \beta_{i^*} \right) \cdot \frac{1}{4} \right)^4 \\
&\leq \left(\frac{32}{10} \cdot \frac{e\beta_{i^*}}{4} \right)^4
\end{aligned}$$

Taking the union bound across all possible $\beta_{i^*}n$ bins, we have the following inequality

$$\begin{aligned}
&\Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \leq (\beta_{i^*}n) \cdot \left(\frac{32}{10} \cdot \frac{e\beta_{i^*}}{4} \right)^4 \\
&\implies \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \leq (18 \log n) \cdot \left(\frac{32}{10} \cdot \frac{18e \log n}{4n} \right)^4 \\
&\implies \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \leq \frac{1}{2n^2} \text{ (since } n \geq n_0)
\end{aligned} \tag{4}$$

Putting together equations 1, 3, and 4, we get

$$\begin{aligned}
\Pr(v_{i^*+4} > 0) &\leq \frac{1}{2n^2} + \frac{1}{2n^2} \\
&\leq \frac{1}{n^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\Pr \left(\text{Max. Load} \geq \frac{\log \log n}{\log k} + 15 \right) &= \Pr(v_{i^*+4} > 0) \\
&\leq \frac{1}{n^2}
\end{aligned}$$

□

From Lemma 1 and Lemma 2, we immediately arrive at Theorem 1.

□

4 Analysis of FirstDiff[d] when $m \gg n$

Theorem 2. Use FirstDiff[d] to allocate m balls into n bins. When $m \geq c_1 \frac{2^{\Theta(d)}}{\Theta(d)} n \log n$, where c_1 is an appropriately chosen constant, it takes d probes on average to place every ball on expectation. Furthermore, for an absolute constant c ,

$$\Pr \left(\text{Max. load of any bin} > \frac{m}{n} + \frac{\log \log n}{\Theta(d)} + c \log \log \log n \right) \leq c(\log \log n)^{-4}.$$

Proof. First we show that the average number of probes per ball is d . We then show the maximum load bound holds.

4.1 Proof of Number of Probes

The main difficulty with analysing the expected number of probes comes from the fact that the number of probes needed for each ball depends on where each of the previous balls were placed. Intuitively, if all the previous balls were placed such that each bin has the same number of balls, the number of probes will be $2^{\Theta(d)}$. On the other hand, if a significant number of bins are at different load levels, then, the ball will be placed with very few probes. One might hope to prove that the system always displays a variety of loads, but unfortunately, the system (as we verified experimentally) oscillates between being very evenly loaded and otherwise. Therefore, we have to take a slightly more nuanced approach that takes into account that the number of probes cycles between high (i.e. as high as $2^{\Theta(d)}$) when the loads are even and as low as 2 when there is more variety in the load.

Lemma 3. When $m \geq c_1 \frac{2^{\Theta(d)}}{\Theta(d)} n \log n$, where c_1 is an appropriately chosen constant, it takes at most md probes on expectation to place the m balls in n bins using FirstDiff[d].

Proof. Let the maximum number of probes allowed per ball using FirstDiff[d] be k , i.e. $k = 2^{\Theta(d)}$. Let $m = \mathcal{L}n$ balls be placed into n bins; we assume $\mathcal{L} \geq c_1 \frac{2^{\Theta(d)}}{\Theta(d)} \log n$. Throughout this proof, we will assume that the maximum load of any bin is at most $m/n + O(\log n)$, which holds with high probability owing to Lemma 10. The low probability event that the maximum load exceeds $m/n + O(\log n)$ will contribute very little to the overall expectation because the probability that any ball exceeds a height of $m/n + O(\log n)$ will be an arbitrarily small inverse polynomial in n . Therefore, such a ball will contribute $o(1)$ to the overall expected number of probes even when we liberally account k probes for each such ball (as long as $k \ll n$).

Consider the balls ordered according to the sequence in which they are placed. Let us define a *placement* of balls as an assignment of a unique ID (ℓ, x) to each ball with both $\ell > 0$ and $0 \leq x \leq n - 1$ being integers such that

1. for any ℓ and any $0 \leq x < x' \leq n - 1$, ball (ℓ, x) must precede ball (ℓ, x') , and
2. for all $1 \leq \ell < \ell'$ and an x such that ball (ℓ', x) exists, ball (ℓ, x) must exist and precede ball (ℓ', x) .

Let $C_{\ell, x}$ be the event that all balls (ℓ'', x'') preceding the ball with ID (ℓ, x) (including (ℓ, x)) extended the number of balls at level ℓ'' from x'' to $x'' + 1$. Similarly, we define $\hat{C}_{\ell, x}$ to be the event that all balls (ℓ'', x'') strictly preceding the ball with ID (ℓ, x) (i.e., excluding (ℓ, x)) extended the number of balls at level ℓ'' from x'' to $x'' + 1$.

We will be concerned with the set of all placements \mathcal{P} such that the maximum value of ℓ is at most $m/n + O(\log n)$.

Consider an arbitrary placement $P \in \mathcal{P}$. In a slight abuse of notation, we denote $C(P)$ to be the event that all balls (ℓ, x) in P extended level ℓ from x balls to $x + 1$ balls. Notice that it suffices to prove our bound on the expected number of probes for this arbitrary placement P conditioned on $C(P)$. Notice that any placing of balls using FirstDiff[d] that respects the height limit of $m/n + O(\log n)$ falls into an appropriately chosen event $P|C(P)$. Therefore, we can apply the law of total expectation over all possible

$P \in \mathcal{P}$ (conditioned on $C(P)$) and conclude that the bound holds over all $P \in \mathcal{P}$. To see this, suppose we upper bound $E[\text{number of probes} | (P|C(P))]$ by some U . Then, notice that by the law of total expectation

$$\begin{aligned} E[\text{number of probes}] &= \sum_{P \in \mathcal{P}} E[\text{number of probes} | (P|C(P))] \cdot \Pr(P|C(P)) \\ &= \sum_{P \in \mathcal{P}} U \cdot \Pr(P|C(P)) \\ &= U \cdot \sum_{P \in \mathcal{P}} \Pr(P|C(P)) \\ &= U \end{aligned}$$

Thus, for the rest of the proof, we will assume a particular (but arbitrarily chosen) placement P .

Let us consider a level to be complete if n balls are present on that level, i.e. there is one ball at that height in every bin. We call a level incomplete if fewer than n balls are present on that level. Let the number of incomplete levels below the \mathcal{L}^{th} level be D .

We begin with a focus on complete levels. We pick an arbitrary complete level ℓ and upper bound the number of probes taken by all balls with IDs (ℓ, x) , $0 \leq x \leq n-1$.

We then upper bound the value of D . Once we have this upper bound on the number of probes for all balls from a given level and the value of D , we can upper bound the total number of probes to place balls in each complete level. We then bound the total number of probes in the incomplete levels and derive the required average value.

Let $X_{\ell,x}$ be a random variable denoting the number of probes required to place a ball with ID (ℓ, x) given $\hat{C}_{\ell,x}$. Here $1 \leq \ell \leq \mathcal{L} - D$ and $0 \leq x \leq n-1$. Note that $X_{\ell,x}$ is not conditioned on ball (ℓ, x) extending level ℓ from x to $x+1$ (unless the condition $C_{\ell,x}$ is applied explicitly).

Lemma 4. $E[X_{\ell,x} | C_{\ell,x}] = E[X_{\ell,x}]$, for any level ℓ , $1 \leq \ell \leq \mathcal{L} - D$ and x , $0 \leq x \leq n-1$.

Before we prove Lemma 4, we take a short detour and prove the main idea on which the proof of Lemma 4 hinges by way of a simpler experiment (that is unrelated to our context other than to elicit the intuition behind our Proof of Lemma 4). Consider a bag of B balls out of which r balls are red and b balls are blue. The remaining $c = B - r - b$ balls are colourless. The experiment is to pick balls sequentially with replacement until the first coloured ball is picked. Let X denote the number of times we pick balls and let R be the condition that we picked a red ball to terminate the experiment.

Lemma 5. $E[X|R] = E[X]$.

Proof. Consider the event $X = t$ for some $t > 0$.

$$\begin{aligned} \Pr(X = t) \cdot \Pr(R) &= (c/B)^{t-1} \frac{r+b}{B} \cdot \frac{r}{r+b} \\ &= (c/B)^{t-1} \frac{r}{B} \\ &= \Pr((X = t) \cap R) \end{aligned}$$

Thus, events $X = t$ and R are independent.

Therefore,

$$\begin{aligned} E[X|R] &= \sum_t t \cdot \Pr((X = t)|R) \\ &= \sum_t t \cdot \Pr(X = t) \\ &= E[X] \end{aligned}$$

□

The key takeaway here is that the number of times we need to pick a ball to get a coloured ball does not depend on whether the coloured ball picked is red or blue. This translates to the idea that the number of probes required to place a ball with ID (ℓ, x) given $\hat{C}_{\ell, x}$ is independent of where that ball is actually placed. With the intuition gained from Lemma 5, we now proceed with the proof of Lemma 4.

Proof of Lemma 4. Consider an arbitrary level ℓ , $1 \leq \ell \leq \mathcal{L} - D$ and x , $0 \leq x \leq n - 1$. We show that the events $X_{\ell, x} = b$, for some arbitrary non-negative integer b , and $C_{\ell, x}$ are independent events. We do this by introducing two more random variables that we condition on, namely T which represents which technique of ball placement was used, and LB which represents the load of the first bin probed. Technique of ball placement refers to the following:

1. All probes were made to bins of the same load.
2. One or more probes were made to bins of larger load followed by a probe to a bin of lesser load.
3. One or more probes were made to bins of lesser load followed by a probe to a bin of larger load.

Let $T = 1$, $T = 2$, and $T = 3$ represent the events that the corresponding techniques were used.

At the time the $(\ell, x)^{\text{th}}$ ball is going to be placed, the configuration of balls and bins will be such that there may be bins with varying loads. Consider the bins sorted in non-increasing order from left to right. Define a plateau to be a maximal set of bins that have the same number of balls. Now, there will be several plateaus of bins having the same load grouped together. Let the plateaus be numbered $1, 2, \dots, F$ from maximum load to minimum load. Let $\alpha_1, \alpha_2, \dots, \alpha_F$ denote the number of bins belonging to the corresponding plateau. Now let $LB = 1, LB = 2, \dots, LB = F$ be the events that the first probe for the ball is made to some bin in the corresponding plateau.

Now we show $\forall 1 \leq v_1 \leq 3, 1 \leq v_2 \leq F$ that

$$\begin{aligned} & \Pr((X_{\ell, x} = b \bigcap C_{\ell, x}) | T = v_1, LB = v_2) \\ &= \Pr(X_{\ell, x} = b | T = v_1, LB = v_2) \cdot \Pr(C_{\ell, x} | T = v_1, LB = v_2). \end{aligned} \quad (5)$$

Let the bins of level $\ell - 1$ belong to the r^{th} plateau. We consider each of the events related to T separately.

Case $v_1 = 1$: Since all probes are made to the same level, the only reasonable value for b is k . When v_2 is chosen such that $1 \leq v_2 \leq r - 1$ or $r + 1 \leq v_2 \leq F$,

$$\Pr(X_{\ell, x} = b \bigcap C_{\ell, x} | T = 1, LB = v_2) = 0$$

$$\begin{aligned} \Pr(X_{\ell, x} = b | T = 1, LB = v_2) \cdot \Pr(C_{\ell, x} | T = 1, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^b \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, Equation 5 is satisfied.

When $v_2 = r$,

$$\Pr(X_{\ell, x} = b \bigcap C_{\ell, x} | T = 1, LB = v_2) = \left(\frac{\alpha_{v_2}}{n}\right)^b$$

$$\begin{aligned} \Pr(X_{\ell, x} = b | T = 1, LB = v_2) \cdot \Pr(C_{\ell, x} | T = 1, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^b \cdot 1 \\ &= \left(\frac{\alpha_{v_2}}{n}\right)^b \end{aligned}$$

Therefore, Equation 5 is satisfied.

Case 2: $v_1 = 2$. Whenever $b < 2$ or $b > k$, when conditioning on $T = 2$, both sides of the equation are 0. Therefore, the following probabilities are arrived at considering $2 \leq b \leq k$. When $v_1 = 2$, the ball will be placed in the last probed bin. Therefore, in order for x to increase by 1, the last bin probed must be of level $\ell - 1$, i.e. it must belong to the r^{th} plateau.

When $1 \leq v_2 \leq r - 1$,

$$\begin{aligned}\Pr(X_{\ell,x} = b \bigcap C_{\ell,x} | T = 2, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\alpha_r}{n}\right) \\ \Pr(X_{\ell,x} = b | T = 2, LB = v_2) \cdot \Pr(C_{\ell,x} | T = 2, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=v_2+1}^F \alpha_j}{n}\right) \cdot \left(\frac{\alpha_r}{\sum_{j=v_2+1}^F \alpha_j}\right) \\ &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\alpha_r}{n}\right)\end{aligned}$$

Therefore, Equation 5 is satisfied.

When $r \leq v_2 \leq F$,

$$\begin{aligned}\Pr(X_{\ell,x} = b \bigcap C_{\ell,x} | T = 2, LB = v_2) &= 0 \\ \Pr(X_{\ell,x} = b | T = 2, LB = v_2) \cdot \Pr(C_{\ell,x} | T = 2, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=v_2+1}^F \alpha_j}{n}\right) \cdot 0 \\ &= 0\end{aligned}$$

Therefore, Equation 5 is satisfied.

Case 3: $v_1 = 3$. Whenever $b < 2$ or $b > k$, when conditioning on $T = 3$, both sides of the equation are 0. Therefore, the following probabilities are arrived at considering $2 \leq b \leq k$. When $v_1 = 3$, the ball will be placed in one of the initially probed bins of lower load. Therefore, in order for x to increase by 1, the first $b - 1$ bins probed must be of level $\ell - 1$, i.e. they must belong to the r^{th} plateau.

When $1 \leq v_2 \leq r - 1$ or $r + 1 \leq v_2 \leq F$,

$$\begin{aligned}\Pr(X_{\ell,x} = b \bigcap C_{\ell,x} | T = 3, LB = v_2) &= 0 \\ \Pr(X_{\ell,x} = b | T = 3, LB = v_2) \cdot \Pr(C_{\ell,x} | T = 3, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=1}^{v_2-1} \alpha_j}{n}\right) \cdot 0 \\ &= 0\end{aligned}$$

Therefore, Equation 5 is satisfied.

When $v_2 = r$,

$$\begin{aligned}\Pr(X_{\ell,x} = b \cap C_{\ell,x} | T = 3, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=1}^{v_2-1} \alpha_j}{n}\right) \\ \Pr(X_{\ell,x} = b | T = 3, LB = v_2) \cdot \Pr(C_{\ell,x} | T = 3, LB = v_2) &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=1}^{v_2-1} \alpha_j}{n}\right) \cdot 1 \\ &= \left(\frac{\alpha_{v_2}}{n}\right)^{b-1} \left(\frac{\sum_{j=1}^{v_2-1} \alpha_j}{n}\right)\end{aligned}$$

Therefore, Equation 5 is satisfied.

Therefore, for all values of $1 \leq v_1 \leq 3$, $1 \leq v_2 \leq F$,
 $\Pr((X_{\ell,x} = b \cap C_{\ell,x}) | T = v_1, LB = v_2)$
 $= \Pr(X_{\ell,x} = b | T = v_1, LB = v_2) \cdot \Pr(C_{\ell,x} | T = v_1, LB = v_2).$
That is to say that for all values of $1 \leq v_1 \leq 3$ and $1 \leq v_2 \leq F$, $X_{\ell,x} = b$ and $C_{\ell,x}$ are independent events.
Because the two events are independent, we have

$$\begin{aligned}E[X_{\ell,x} | C_{\ell,x}] &= \sum_{b=0}^{\infty} b \Pr(X_{\ell,x} = b | C_{\ell,x}) \\ &= \sum_{b=0}^{\infty} \sum_{v_1=1}^3 \sum_{v_2=1}^F b \Pr(X_{\ell,x} = b | C_{\ell,x}, T = v_1, LB = v_2) \\ &= \sum_{b=0}^{\infty} \sum_{v_1=1}^3 \sum_{v_2=1}^F b \Pr(X_{\ell,x} = b | T = v_1, LB = v_2) \\ &= E[X_{\ell,x}]\end{aligned}$$

□

We now upper bound the expected number of probes required by a ball in an arbitrary configuration to be placed. For a given level ℓ and number of balls at that level x , we define the canonical configuration for (ℓ, x) to be the configuration such that

1. There are no balls at levels greater than ℓ .
2. There are x balls at level ℓ .
3. All levels less than ℓ are complete.

Define random variable, $Y_{\ell,x}$ = the number of probes required to place a ball in the canonical configuration for (ℓ, x) .

Lemma 6. *For any ball (ℓ, x) , $1 \leq \ell \leq \mathcal{L} - D$, and $0 \leq x \leq n - 1$, $E[X_{\ell,x}] \leq E[Y_{\ell,x}]$.*

Proof. Sort the bins in non-increasing order from left to right. In an arbitrary configuration, there may exist bins at several heights h_1, h_2, \dots, h_F . Let there be $\alpha_1, \alpha_2, \dots, \alpha_F$ such bins. Now, all that is required to place the ball is if your first probe is to some bin of height h_j , then you must make another probe to one of the $n - \alpha_j$ bins. Contrast this with the canonical configuration, where several bins of heights h_1, h_2, \dots, h_k have been flattened to height h_k and the remaining bins have been flattened (filled with balls if required) to height $h_k - 1$. After the initial probe, the probability of probing another bin of a different height is reduced in this case. Hence the expected number of probes required to place a ball in the canonical configuration is more than in an arbitrary configuration. \square

We now provide a bound on the expected number of probes to place n balls of the same level in the canonical case.

Lemma 7. $\sum_{x=0}^{n-1} E[Y_{\ell,x}] = O(n \log k)$.

Proof. For a given canonical configuration (ℓ, x) , the expected number of probes is upper bounded as follows

$$\begin{aligned} E[Y_{\ell,x}] &\leq \frac{x}{n} \frac{n}{n-x} + \frac{n-x}{n} \frac{n}{x} \\ &\leq \frac{x}{n-x} + \frac{n-x}{x} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} E[Y_{\ell,x}] &\leq \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{n-x}{x} + \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{x}{n-x} \\ &= n \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{1}{x} + n \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{1}{n-x} - \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{n-x}{n-x} \\ &\leq n \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{1}{x} + n \sum_{y=n-\frac{n}{k}}^{\frac{n}{k}} \frac{1}{y} \\ &= 2n \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} \frac{1}{x} \\ &\approx 2n \left(\log \left(n - \frac{n}{k} \right) - \log \left(\frac{n}{k} \right) \right) \\ &= 2n \log(k-1) \\ &\leq 2n \log k \end{aligned}$$

For the first $\frac{n}{k}$ and last $\frac{n}{k} - 1$ balls, let us give away the maximum number of probes, i.e. $E[Y_{\ell,x}] = k$ for $0 \leq x \leq \frac{n}{k} - 1$ and $n - \frac{n}{k} + 1 \leq x \leq n - 1$. Therefore, the expected total number of probes

$$\begin{aligned} \sum_{x=0}^{n-1} E[Y_{\ell,x}] &= \sum_{x=0}^{\frac{n}{k}-1} E[Y_{\ell,x}] + \sum_{x=\frac{n}{k}}^{n-\frac{n}{k}} E[Y_{\ell,x}] + \sum_{x=n-\frac{n}{k}+1}^{n-1} E[Y_{\ell,x}] \\ &\leq k \frac{n}{k} + 2n \log k + k \left(\frac{n}{k} - 1 \right) \\ &= (2 + o(1))(n \log k). \end{aligned}$$

\square

Now that we have the expected number of probes required to place every ball in a given complete level, we need to calculate how many such complete levels are present. Towards this, we bound the value of D , i.e. the number of incomplete levels below the \mathcal{L}^{th} level.

Lemma 8. $D = O(\log n)$ with high probability.

Proof. Assume for the sake of contradiction that $D = \omega(\log n)$ with high probability. Now, we derive the following claim from our assumption.

Claim. Assuming $D = \omega(\log n)$, the total number of balls in incomplete levels is in $\omega(n \log n)$.

Proof. This can be seen by the fact that below level $\mathcal{L} - D$, $(\mathcal{L} - D - 1)n$ balls have been placed. That leaves $\mathcal{L}n - (\mathcal{L} - D - 1)n = (D + 1)n = \omega(n \log n)$ balls left to be placed in higher levels. \square

Once all $\mathcal{L}n$ balls have been placed, sort the bins in non-increasing order of load from left to right. Consider the rightmost bin. Since the level of this bin is still at $(\mathcal{L} - D)$, not one of the $\omega(n \log n)$ balls should have been placed in it. In order for this to happen, none of the probes made for at least $\omega(n \log n) - (n - 1)$ balls could have been made to this bin. This is because once at most $(n - 1)$ balls were added to level $\mathcal{L} - D + 1$, even if a single probe was made to the given bin, it would result in a ball being placed in that bin. The probability that no probe is made to this bin is at most $(1 - \frac{1}{n})^{\omega(n \log n) - n} \approx e^{-\omega(\log n) + 1} = e \cdot n^{-\omega(1)}$. This implies that with high probability, some probes from the $\omega(n \log n) - n$ balls may have been made to this bin, resulting in this bin being filled and our assumption being contradicted. Hence the lemma is proved. \square

We know that $\mathcal{L} = \frac{m}{n}$. Now, consider all complete levels $1 \leq \ell \leq \mathcal{L} - O(\log n)$. We know that for each level, the expected total number of probes required by each ball at that level is in $O(n \log k)$ by Lemma 4, Lemma 6 and Lemma 7. Therefore the expected number of probes required to place every ball in the complete levels is in $O((\mathcal{L} - O(\log n))n \log k)$.

Now consider the levels ℓ that are incomplete, with values of ℓ ranging from $\mathcal{L} - O(\log n) \leq \ell \leq \mathcal{L} + \frac{\log \log n}{\log k} + O(1)$. The number of probes each ball in these levels takes is (for many balls loosely) upper bounded by k and we liberally account for the number of probes per ball as k .

Then the number of probes taken by all balls in incomplete levels $= O(k \cdot n \log n) = c_1 k \cdot n \log n$. Because $m \geq c_1 \frac{2^{\Theta(d)}}{\Theta(d)} n \log n$ and $k = 2^{\Theta(d)}$, the previous value is upper bounded by $\frac{m}{n} \log k$. Therefore the total number of probes taken by all balls in the incomplete levels is $= O(\mathcal{L}n \log k)$.

Therefore $E[\text{Total number of probes for every ball at every level}] = O(\mathcal{L}n \log k)$. Therefore on average, $E[\text{Number of probes per ball}] = O(\log k) = O(\log(2^{\Theta(d)})) = d$, for an appropriately chosen constant. \square

4.2 Proof of Maximum Load

Lemma 9. For any m , for an absolute constant c ,

$$\Pr \left(\text{Max. load of any bin} > \frac{m}{n} + \frac{\log \log n}{\Theta(d)} + c \log \log \log n \right) \leq c(\log \log n)^{-4}.$$

Proof. This proof follows along the lines of that of Theorem 2 from [12]. In order to prove Lemma 9, we make use of a theorem from [8] which gives us an initial, loose, bound on the gap G^t between the maximum load and average load for an arbitrary m . We then use a lemma to tighten this gap. We use one final lemma to show that if this bound on the gap holds after all m balls are placed, then it will hold at any time prior to that.

First, we establish some notation. Let k be the maximum number of probes permitted to be made per ball by FirstDiff[d], i.e. $k = 2^{\Theta(d)}$. After placing nt balls, let us define the *load vector* X^t as representing the difference between the load of each bin and the average load (as in [2, 8]). Without loss of generality we order the individual values of the vector in non-increasing order of load difference, i.e. $X_1^t \geq X_2^t \geq \dots \geq X_n^t$. So X_i^t is the load in the i^{th} most loaded bin minus t . For convenience, denote X_1^t (i.e. the gap between the heaviest load and the average) as G^t .

Initial Bound on Gap

We now give an upper bound for the gap between the maximum loaded bin and the average load after placing some arbitrary number of balls nt . In other words, we show $\Pr(G^t \geq x)$ is negligible for some x . This x will be our initial bound on the gap G^t .

Lemma 10. *For arbitrary constant c , after placing an arbitrary nt balls into bins under FirstDiff[d], there exist some constants a and b such that $\Pr(G^t \geq \frac{c \log n}{a}) \leq \frac{bn}{n^c}$. Thus there exists a constant λ that gives $\Pr(G^t \geq \lambda \log n) \leq \frac{1}{n^c}$ for a desired c value.*

In order to prove Lemma 10, we need two additional facts. The first is the following basic observation:

Lemma 11. *FirstDiff[d] is majorized by Greedy[2] when $d \geq 2$.*

Proof. Let the load vectors for FirstDiff[d] and Greedy[2] after t balls have been placed using the respective algorithms be u^t and v^t respectively. Now we follow the standard coupling argument (refer to Section 5 in [2] for an example). Couple FirstDiff[d] with Greedy[2] by letting the bins probed by Greedy[2] be the first 2 bins probed by FirstDiff[d]. We know that FirstDiff[d] makes at least 2 probes when $d \geq 2$. It is clear that FirstDiff[d] will always place a ball in a bin with load less than or equal to that of the bin chosen by Greedy[2]. This ensures that if majorization was preserved prior to the placement of the ball, then the new load vectors will continue to preserve majorization; again, see [2] for a detailed example. Initially, u^0 is majorized by v^0 since both vectors are the same. Using induction, it can be seen that if u^t is majorized by v^t at the time the t^{th} ball was placed, it would continue to be majorized at time $t+1$, $0 \leq t \leq m-1$. Therefore, FirstDiff[d] is majorized by Greedy[2] when $d \geq 2$. \square

The other fact is the following theorem about Greedy[d] taken from [8] (used similarly in [12] as Theorem 3).

Theorem 3. [8] *Let Y^t be the load vector generated by Greedy[d]. Then for every $d > 1$ there exist positive constants a and b such that for all n and all t ,*

$$E \left(\sum_i e^{a|Y_i^t|} \right) \leq bn.$$

We are now ready to prove Lemma 10.

Proof of Lemma 10. Combining Lemma 11 with Theorem 3 tells us that, if X^t is the load vector generated by FirstDiff[d],

$$E \left(\sum_i e^{a|X_i^t|} \right) \leq bn.$$

Clearly, $\Pr(G^t \geq \frac{c \log n}{a}) = \Pr(e^{aG^t} \geq n^c)$. Observe that $\sum_i e^{a|X_i^t|} \geq e^{aG^t}$. Then

$$\begin{aligned} \Pr(G^t \geq \frac{c \log n}{a}) &= \Pr(e^{aG^t} \geq n^c) \\ &\leq \frac{E[e^{aG^t}]}{n^c} \quad (\text{by Markov's inequality}) \\ &\leq \frac{bn}{n^c} \quad (\text{by Theorem 3 and Lemma 11}) \end{aligned}$$

and the theorem is proved. \square

Reducing the Gap

Lemma 10 gives an initial bound on G^t of order $\log n$. The next step is to reduce it to our desired gap value. For this reduction, we use a modified version of Lemma 2 from [12], with a similar but more involved proof. We now give the modified lemma and prove it.

Lemma 12. *For every k , there exists a universal constant γ such that the following holds: for any t, ℓ, L such that $1 \leq \ell \leq L \leq n^{\frac{1}{4}}$, $L = \Omega(\log \log n)$ and $\Pr(G^t \geq L) \leq \frac{1}{2}$,*

$$\Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma) \leq \Pr(G^t \geq L) + \frac{16bL^3}{e^{at}} + \frac{1}{n^2},$$

where a and b are the constants from Theorem 3.

Proof. This proof consists of many steps. We first observe that Lemma 12 follows directly from Lemma 10 for sufficiently small n . We then use layered induction to bound the proportion of bins of each size for larger n . This in turn allows us to compute our desired bound on the probability of a large gap occurring.

Proof of Lemma 12 for smaller values of n

Define n_1 to be the minimum value of n such that $L \geq 2$ (recall $L = \Omega(\log \log n)$). Define n_2 to be the minimum value of n such that

$(18 \log n) * \left(\frac{18en^{\frac{1}{4}} \log n}{n} \right)^4 \leq \frac{1}{2n^2}$. Define n_3 to be the minimum value of n such that $n \geq 54 \log n$. Define absolute constant $n_0 = \max(n_1, n_2, n_3)$.

Notice that, when $n \leq n_0$, Lemma 10 implies that Lemma 12 holds with $\gamma = O(\log n_0)$. If $n \leq n_0$, then

$$\Pr(G^{t+\ell} \geq \log \log n + \ell + \gamma) \leq \Pr(G^{t+L} \geq \gamma).$$

Consider the right hand side of Lemma 12.

$$\Pr(G^t \geq L) + \frac{16bL^3}{\exp(a\ell)} + \frac{1}{n^2} \geq n^{-2},$$

so it will be sufficient to prove the inequality

$$\Pr(G^{t+L} \geq \gamma) \leq n^{-2}.$$

Since there are no conditions on t in Lemma 10, we may rewrite it as

$$\Pr(G^{t+L} \geq \lambda \log n) \leq n^{-c}.$$

Let $c = 2$ and compute the constant λ accordingly. Set $\gamma = \lambda \log n_0 \geq \lambda \log n$. Then

$$n^{-2} \geq \Pr(G^{t+L} \geq \lambda \log n) \geq \Pr(G^{t+L} \geq \gamma),$$

and we are done.

Rewriting initial probability inequality

We now prove Lemma 12 assuming $n > n_0$. Start by rewriting the probability in terms of $\Pr(G^t \geq L)$.

$$\begin{aligned} \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma) &= \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma | G^t \geq L) \Pr(G^t \geq L) \\ &\quad + \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma | G^t < L) \Pr(G^t < L) \\ &\leq \Pr(G^t \geq L) + \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma | G^t < L) \end{aligned}$$

To prove the theorem, then, it is enough to show that

$$\Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + \gamma | G^t < L) \leq \frac{16bL^3}{e^{at}} + \frac{1}{n^2}.$$

Bins' loads

Define v_i to be the fraction of bins of load at least $t + L + i$ after $(t + L)n$ balls are placed. Let us set $i^* = \frac{\log \log n}{\log k} + \ell$ and set $\gamma = 4$. Using this new notation, we want to show that $\Pr(G^{t+L} \geq i^* + 4 | G^t < L)$ is negligible. This can be thought of as showing that the probability of the fraction of bins of load at least $t + L + i^* + 4$ exceeding 0 after $(t + L)n$ balls are placed, conditioned on the event that $G^t < L$, is negligible.

Suppose we have a non-increasing series of numbers $\beta_0, \beta_1, \dots, \beta_i, \dots$ that are upper bounds for $v_0, v_1, \dots, v_i, \dots$. Then we know that

$$\begin{aligned} \Pr(v_{i^*+4} > 0) &= \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) \Pr(v_{i^*} \leq \beta_{i^*}) + \Pr(v_{i^*+4} > 0 | v_{i^*} > \beta_{i^*}) \Pr(v_{i^*} > \beta_{i^*}) \\ &\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) + \Pr(v_{i^*} > \beta_{i^*}) \\ &\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}) + \sum_{j=\ell+1}^{i^*} \Pr(v_j > \beta_j | v_{j-1} \leq \beta_{j-1}) + \Pr(v_\ell > \beta_\ell) \end{aligned}$$

Conditioning both sides on $G^t < L$, we have

$$\begin{aligned} \Pr(v_{i^*+4} > 0 | G^t < L) &\leq \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}, G^t < L) \\ &\quad + \sum_{i=\ell+1}^{i^*} \Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) \\ &\quad + \Pr(v_\ell > \beta_\ell | G^t < L) \end{aligned} \tag{6}$$

It remains to find appropriate β_i values. We use a layered induction approach to show that v_i 's don't exceed the corresponding β_i 's with high probability. This then allows us to upper bound each of the 3 components of equation 6.

Base case of layered induction

In order to use layered induction, we need a base case. Let us set $\beta_\ell = \frac{1}{8L^3}$, for the ℓ in the statement of the theorem. Now,

$$\begin{aligned} \Pr(v_\ell > \beta_\ell | G^t < L) &= \frac{\Pr((v_\ell > \frac{1}{8L^3}) \cap (G^t < L))}{\Pr(G^t < L)} \\ &\leq 2 \cdot \Pr(v_\ell > \frac{1}{8L^3}) \text{ (since, by the statement of the theorem } \Pr(G^t < L) \geq \frac{1}{2}) \\ &\leq 2 \cdot \frac{8bL^3}{e^{a\ell}} \text{ (applying Markov's inequality and using Theorem 3)} \\ &\leq \frac{16bL^3}{e^{a\ell}} \end{aligned}$$

Therefore we have the third term of Equation 6 bounded:

$$\Pr(v_\ell > \beta_\ell | G^t < L) \leq \frac{16bL^3}{e^{a\ell}} \tag{7}$$

Recurrence relation for layered induction

We now define the remaining β_i values recursively. Note that for all $i \geq \ell$, $\beta_i \leq \beta_\ell$. Let u_i be defined as the number of balls of height at least $t + L + i$ after $(L + t)n$ balls are placed.

Initially there were nt balls in the system. Then we threw another nL balls into the system. Remember that $t + L$ is the average load of a bin after nL balls are further placed. Because we condition on $G^t < L$, we have it that any ball of height i , $i \geq 1$, must have been one of the nL balls placed.

Therefore the number of bins of load $t + L + i + 1$ after $(t + L)n$ balls are placed is upper bounded by the number of balls of height at least $t + L + i + 1$. So $v_{i+1}n \leq u_{i+1}$. In order to upper bound v_{i+1} , we can upper bound u_{i+1} .

Recall the algorithm places a ball in a bin of load $t + L + i$ if it probes k times and sees a bin of load $t + L + i$ each time; or if it probes $j < k$ times and sees a bin of load $t + L + i$ each time, then probes a bin of load $\geq t + L + i + 1$; or if it probes $j < k$ times and sees a bin of load at least $t + L + i + 1$ each time (where the load of the bin probed each time is the same), then probes a bin of load $t + L + i$. Thus the probability that a ball will end up at height at least $t + L + i + 1$ is

$$\begin{aligned}
&\leq \beta_i^k + \beta_i \beta_{i+1} (1 + \beta_i + \beta_i^2 + \dots + \beta_i^{k-2}) + \beta_i \beta_{i+1} (1 + \beta_{i+1} + \beta_{i+1}^2 + \dots + \beta_{i+1}^{k-2}) \\
&\leq \beta_i^k + \beta_i \beta_{i+1} \left(\frac{1 - \beta_i^{k-1}}{1 - \beta_i} + \frac{1 - \beta_{i+1}^{k-1}}{1 - \beta_{i+1}} \right) \\
&\leq \beta_i^k + \beta_i \beta_{i+1} \left(2 * \frac{1}{1 - \beta_i} \right) \\
&\leq \beta_i^k + \frac{2\beta_{i+1}}{8L^3 - 1}
\end{aligned}$$

Let $v_{i+1}(f)$ be the fraction of bins with load at least $t + i + 1$ after the $tn + f^{\text{th}}$, $1 \leq f \leq nL$, ball is placed in a bin.

Let $f^* = \min[\arg \min_f v_{i+1}(f) > \beta_{i+1}, nL]$, i.e. f^* is the first f such that $v_{i+1}(f) > \beta_{i+1}$ or nL if there is no such f . By our preceding argument, the probability that $f^* < nL$ is bounded by the probability that a binomial random variable $B(nL, \leq \beta_i^k + \frac{2\beta_{i+1}}{8L^3-1})$ is greater than $\beta_{i+1}nL$.

Fix

$$\beta_{i+1} = 2L \frac{8L^3 - 1}{8L^3 - 4L - 1} \beta_i^k \geq \frac{2nL(\beta_i^k + \frac{2\beta_{i+1}}{8L^3-1})}{n}.$$

Then using a Chernoff bound, we can say that with high probability, $f^* = nL$ or $v_{i+1} \leq \beta_{i+1}$, so long as $e^{-\frac{(\beta_i^k + \frac{2\beta_{i+1}}{8L^3-1})}{3}} = O(\frac{1}{n^c})$ for some constant $c \geq 1$.

Now, so long as $\beta_{i+1} \geq \frac{18 \log n}{n}$, $e^{-\frac{(\beta_i^k + \frac{2\beta_{i+1}}{8L^3-1})}{3}} = O(\frac{1}{n^c})$. In other words, this upper bound holds for the placement of all $nt + nL$ balls.

We now show that according to the previous recurrence relation, β_{i^*} dips below $\frac{18 \log n}{n}$. We later propose a modified recurrence relation which sets the value of β_i to the maximum of the value of obtained from the recurrence and $\frac{18 \log n}{n}$. This ensures that $\beta_{i^*} = \frac{18 \log n}{n}$. This upper bound will be used later in the argument. We have, from the value of β_ℓ and the above discussion,

$$\begin{aligned}
\log \beta_\ell &= -3 \log(2L) \text{ and} \\
\log \beta_{i+1} &= k \log \beta_i + \log(2L) + \log\left(\frac{8L^3 - 1}{8L^3 - 4L - 1}\right)
\end{aligned}$$

Solving the recursion for $\log \beta_{\ell + \log \log n}$, we get

$$\begin{aligned}
\log \beta_{\ell + \log \log n} &= \frac{k^{\log \log n} - 1}{k - 1} \log \left(\frac{2L(8L^3 - 1)}{8L^3 - 4L - 1} \right) - 3k^{\log \log n} \log(2L) \\
&\leq k^{\log \log n} \left((-3k + 4) \log(2L) + \log \left(\frac{8L^3 - 1}{8L^3 - 4L - 1} \right) \right) \\
&\leq k^{\log \log n} \left((-6 + 4) \log(2L) + \log \left(\frac{8L^3 - 1}{8L^3 - 4L - 1} \right) \right) \\
&\leq k^{\log \log n} ((-1.5) \log(2L)) \text{ (when } L \geq 2) \\
&\leq 2^{\log \log n} ((-1.5) \log(2L)) \\
&\leq (-1.5)(\log n)
\end{aligned}$$

Therefore, $\beta_{i^*} \leq n^{-1.5}$, when $L \geq 2$. Since $n \geq n_1$, we have $L \geq 2$. Thus $\beta_{i^*} < \frac{18 \log n}{n}$, as desired.

Now, we need to bound $\Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L)$ for all i 's from $\ell + 1$ to i^* . Let us set $\beta_{i+1} = \max(2L \frac{8L^3-1}{8L^3-4L-1} \beta_i^k, \frac{18 \log n}{n})$.

Using the values of β_i generated above, we prove that for all i such that $\ell + 1 \leq i \leq i^*$, $\Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) \leq \frac{1}{n^3}$.

For a given i ,

$$\begin{aligned} \Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) &= \Pr(nv_i > n\beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) \\ &\leq \Pr(u_i > n\beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) \end{aligned}$$

We now upper bound the above inequality using the following idea. Let Y_r be an indicator variable set to 1 when all three of the following conditions are met: (i) the $nt + r^{\text{th}}$ ball placed is of height at least $t + L + i$, (ii) $v_{i-1} \leq \beta_{i-1}$ and (iii) $G^t < L$. Y_r is set to 0 otherwise. Now for all $1 \leq r \leq nL$, the probability that $Y_r = 1$ is upper bounded by $\beta_i^k + \frac{2\beta_{i+1}}{8L^3-1} \leq \frac{8L^3-1}{8L^3-4L-1} \beta_{i-1}^k \leq \frac{\beta_i}{2L}$. Since we condition on $G^t < L$, the number of balls of height at least $t + L$ or more come only from the nL balls placed. Therefore, the probability that the number of balls of height at least $n + L + i$ exceeds β_i is upper bounded by $\Pr(B(nL, \frac{\beta_i}{2L}) > \beta_i)$, where $B(.,.)$ is a Bernoulli trial with given parameters.

According to Chernoff's bound, for $0 < \delta \leq 1$, $\Pr(X \geq (1 + \delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$, where X is the sum of independent Poisson trials and μ is the expectation of X . If we set $\delta = 1$, then we have

$$\begin{aligned} \Pr(v_i > \beta_i | v_{i-1} \leq \beta_{i-1}, G^t < L) &\leq \Pr(B(nL, \frac{\beta_i}{2L}) > \beta_i) \\ &\leq e^{-\frac{n * (\frac{\beta_i}{2})}{3}} \\ &\leq e^{-\frac{n * (\frac{18 \log n}{n})}{6}} \quad (\text{since } \beta_i \geq \frac{18 \log n}{n}, \forall i \leq i^*) \\ &\leq \frac{1}{n^3} \end{aligned}$$

Thus we bound the middle term in Equation 6

$$\begin{aligned} \sum_{j=\ell+1}^{i^*} \Pr(v_j > \beta_j | v_{j-1} \leq \beta_{j-1}, G^t < L) &\leq \frac{\log \log n}{n^3} \\ \implies \sum_{j=\ell+1}^{i^*} \Pr(v_j > \beta_j | v_{j-1} \leq \beta_{j-1}, G^t < L) &\leq \frac{1}{2n^2} \quad (\text{since } n \geq n_1) \end{aligned} \tag{8}$$

Top layers of layered induction

Finally, we need to upper bound the first term in Equation 6,

$\Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}, G^t < L)$. Consider a bin of load at least i^* . We will upper bound the probability that a ball falls into this specific bin. Regardless of how the probes are made for that ball, one of them must be made to that specific bin. Thus we have a formula similar to our original recursion, but with a factor of $1/n$.

Therefore the probability that a ball will fall into that bin is

$$\begin{aligned}
&\leq \frac{1}{n} \beta_i^{k-1} + \frac{1}{n} \beta_{i^*+1} (1 + \beta_{i^*} + \beta_{i^*} + \dots + \beta_{i^*}^{k-2}) + \frac{1}{n} \beta_{i^*+1} (1 + \beta_{i^*+1} + \beta_{i^*+1}^2 + \dots + \beta_{i^*+1}^{k-2}) \\
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + 2\beta_{i^*+1} \left(\frac{1}{1 - \beta_{i^*+1}} \right) \right) \\
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + 2\beta_{i^*} \left(\frac{1}{1 - \beta_{i^*}} \right) \right) \\
&\leq \frac{1}{n} \cdot \left(\beta_{i^*}^{k-1} + \frac{2n}{n - 18 \log n} \beta_{i^*} \right) \\
&\leq \frac{1}{n} \cdot \frac{3n - 18 \log n}{n - 18 \log n} \cdot \beta_{i^*} \text{ (since } k \geq 2 \text{ and } \beta_{i^*} \leq 1) \\
&\leq \frac{4}{n} \cdot \beta_{i^*} \text{ (since } n > n_3)
\end{aligned}$$

Now, we upper bound the probability that 4 balls fall into a given bin of load at least β_{i^*} and then use a union bound over all the bins of height at least β_{i^*} to show that the probability that the fraction of bins of load at least β_{i^*+4} exceeds 0 is negligible.

First, the probability that 4 balls fall into a given bin of load at least β_{i^*} is

$$\begin{aligned}
&\leq \Pr(B(nL, (\frac{4}{n} \cdot \beta_{i^*})) \geq 4) \\
&\leq \binom{nL}{4} \left(\frac{4}{n} \cdot \beta_{i^*} \right)^4 \\
&\leq \left(e \cdot nL \cdot \left(\frac{4}{n} \cdot \beta_{i^*} \right) \cdot \frac{1}{4} \right)^4 \\
&\leq (eL\beta_{i^*})^4
\end{aligned}$$

Taking the union bound across all possible $\beta_{i^*}n$ bins, we have the following inequality

$$\begin{aligned}
&\Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}, G^t < L) \leq (\beta_{i^*}n) \cdot (eL\beta_{i^*})^4 \\
&\implies \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}, G^t < L) \leq (18 \log n) \cdot \left(\frac{18eL \log n}{n} \right)^4 \\
&\implies \Pr(v_{i^*+4} > 0 | v_{i^*} \leq \beta_{i^*}, G^t < L) \leq \frac{1}{2n^2} \text{ (since } n \geq n_2)
\end{aligned} \tag{9}$$

Putting together equations 6, 7, 8, and 9, we get

$$\begin{aligned}
\Pr(v_{i^*+4} > 0 | G^t < L) &\leq \frac{16bL^3}{e^{a\ell}} + \frac{1}{2n^2} + \frac{1}{2n^2} \\
&\leq \frac{16bL^3}{e^{a\ell}} + \frac{1}{n^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + 4 | G^t < L) &= \Pr(v_{i^*+4} > 0 | G^t < L) \\
&\leq \frac{16bL^3}{e^{a\ell}} + \frac{1}{n^2}
\end{aligned}$$

Finally

$$\begin{aligned} \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + 4) &\leq \Pr(G^t \geq L) + \Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + \ell + 4 | G^t < L) \\ &\leq \Pr(G^t \geq L) + \frac{16bL^3}{e^{a\ell}} + \frac{1}{n^2} \end{aligned}$$

Hence Lemma 12 is proved. \square

By Lemma 10, we know that at some arbitrary time t , the gap will be $O(\log n)$ with high probability. Now applying Lemma 12 once with $L = O(\log n)$ and $\ell = O(\log \log n)$ with appropriately chosen constants, we get $\Pr(G^{t+L} \geq \frac{\log \log n}{\log k} + O(\log \log n) + \gamma) \leq O((\log \log n)^{-4})$. Applying the lemma again with $L = O(\log \log n)$ and $\ell = O(\log \log \log n)$ with appropriately chosen constants, we get $\Pr(G^t > \frac{\log \log n}{\log k} + c \log \log \log n) \leq \frac{c}{(\log \log n)^4}$ when time $t = \omega(\log n)$.

We now show that as more balls are placed, the probability that the gap exceeds a particular value increases. This is true by Lemma 4 from [12]:

Lemma 13. [12] *For $t \geq t'$, $G^{t'}$ is stochastically dominated by G^t . Thus $E[G^{t'}] \leq E[G^t]$ and for every z , $\Pr(G^{t'} \geq z) \leq \Pr(G^t \geq z)$.*

Although the setting is different in [12], their proof of Lemma 13 applies here as well. Thus knowing the gap is large when time $t = \omega(\log n)$ with probability $O((\log \log n)^{-4})$, implies that for all values of $t' < t$, the gap exceeds the desired value with at most the same probability. Substituting $k = 2^{\Theta(d)}$ in $\Pr(G^t > \frac{\log \log n}{\log k} + c \log \log \log n) \leq \frac{c}{(\log \log n)^4}$ and modifying the inequality to talk about max. load after m balls have been thrown results in the lemma statement.

Thus concludes the proof of Lemma 9. \square

Putting together Lemma 3 and Lemma 9, we get Theorem 2. \square

5 Lower Bound on Maximum Bin Load

We now provide a lower bound to the maximum load of any bin after using FirstDiff[d] as well as other types of algorithms which use a variable number of probes for Class 1 type algorithms as defined by Vöcking [13]. Class 1 algorithms are those where for each ball, the locations are chosen uniformly and independently at random from the bins available. We first give a general theorem for this type of algorithm and then apply it to FirstDiff[d].

Theorem 4. *Let Alg[k] be any algorithm that places m balls into n bins, where $m \geq n$, sequentially one by one and satisfies the following conditions:*

1. *At most k probes are used to place each ball.*
2. *For each ball, each probe is made uniformly at random to one of the n bins.*
3. *For each ball, each probe is independent of every other probe.*

The maximum load of any bin after placing all m balls using Alg[k] is at least $\frac{m}{n} + \frac{\ln \ln n}{\ln k} - \Theta(1)$ with high probability.

Proof. We show that Greedy[k] is majorized by Alg[k], i.e. Greedy[k] always performs better than Alg[k] in terms of load balancing. Thus any lower bound that applies to the max. load of any bin after using Greedy[k] must also apply to Alg[k].

Let the load vectors for Greedy[k] and Alg[k] after t balls have been placed using the respective algorithms be u^t and v^t respectively. We use induction on the number of balls placed to prove our claim of majorization.

Initially, no ball is placed and by default u^0 is majorized by v^0 . Assume that u^{t-1} is majorized by v^{t-1} . We now use the standard coupling argument to prove the induction hypothesis. For the placement of the t^{th} ball, let $\text{Alg}[k]$ use w_t probes. Couple $\text{Greedy}[k]$ with $\text{Alg}[k]$ by letting the first w_t bins probed by $\text{Greedy}[k]$ be the same bins probed by $\text{Alg}[k]$. $\text{Greedy}[k]$ will always make at least w_t probes and thus possibly makes probes to lesser loaded bins than those probed by $\text{Alg}[k]$. Since $\text{Greedy}[k]$ places a ball into the least loaded bin it finds, it will place a ball into a bin with load at most the same as the one chosen by $\text{Alg}[k]$. Therefore u^t is majorized by v^t . Thus by induction, we see that u^t is majorized by v^t for all $0 \leq t \leq m$. Therefore $\text{Greedy}[k]$ is majorized by $\text{Alg}[k]$.

It is known that the max. load of any bin after the placement of m balls into n bins ($m \geq n$) using $\text{Greedy}[k]$ is at least $\frac{m}{n} + \frac{\ln \ln n}{\ln k} - \Theta(1)$ with high probability [2]. Therefore, the same lower bound also applies to $\text{Alg}[k]$. \square

Now we are ready to prove our lower bound on the max. load of any bin after using $\text{FirstDiff}[d]$.

Theorem 5. *The maximum load of any bin after placing m balls into n bins using $\text{FirstDiff}[d]$ is at least $\frac{m}{n} + \frac{\ln \ln n}{\Theta(d)} - \Theta(1)$ with high probability.*

Proof. We see that $\text{FirstDiff}[d]$ uses at most $2^{\Theta(d)}$ probes and satisfies the requirements of Theorem 4. Thus by substituting $k = 2^{\Theta(d)}$, we get the desired bound. \square

6 Experimental Results

Table 1: Experimental results for the maximum load for n balls and n bins based on 100 experiments for each configuration. Note that the maximum number of probes per ball in $\text{FirstDiff}[d]$, denoted as k , is chosen such that the average number of probes per ball is fewer than d .

n	$d = 2, k = 3$			$d = 3, k = 10$			$d = 4, k = 30$		
	$\text{Greedy}[d]$	$\text{Left}[d]$	$\text{FirstDiff}[d]$	$\text{Greedy}[d]$	$\text{Left}[d]$	$\text{FirstDiff}[d]$	$\text{Greedy}[d]$	$\text{Left}[d]$	$\text{FirstDiff}[d]$
2^8	2...11% 3...87% 4... 2%	2...43% 3...57%	2...81% 3...19%	2...88% 3...12%	2...100%	2...100%	2...100%	2...100%	2...100%
2^{12}	3...99% 4... 1%	3...100%	2...10% 3...90%	2...12% 3...88%	2...96% 3... 4%	2...100%	2...93% 3... 7%	2...100%	2...100%
2^{16}	3...63% 4...37%	3...98% 4... 2%	3...100%	3...100	2...49% 3...51%	2...100%	2...31% 3...69%	2...100%	2...100%
2^{20}	4...100%	3...96% 4... 4%	3...100%	3...100%	3...100%	2...100%	3...100%	2...100%	2...100%
2^{24}	4...100%	3...37% 4...63%	3...100%	3...100%	3...100%	2...100%	3...100%	2...100%	2...100%

We experimentally compare the performance of $\text{FirstDiff}[d]$ with $\text{Left}[d]$ and $\text{Greedy}[d]$ in Table 1. Similar to the experimental results in [13], we perform all 3 algorithms in different configurations of bins and d values. Let k be the maximum number of probes allowed to be used by $\text{FirstDiff}[d]$ per ball. For each value of $d \in [2, 4]$, we choose a corresponding value of k such that the average number of probes required by each ball in $\text{FirstDiff}[d]$ is at most d . For each configuration, we run each algorithm 100 times and note the percentage of times the maximum loaded bin had a particular value. It is of interest to note that $\text{FirstDiff}[d]$, despite

using on average less than d probes per ball, appears to perform better than both **Greedy** $[d]$ and **FirstDiff** $[d]$ in terms of maximum load.

7 Conclusions and Future Work

In this paper, we have introduced a novel algorithm called **FirstDiff** $[d]$ for the well-studied load balancing problem. This algorithm combines the benefits of two prominent algorithms, namely, **Greedy** $[d]$ and **Left** $[d]$. **FirstDiff** $[d]$ generates a maximum load comparable to that of **Left** $[d]$, while being as fully decentralized as **Greedy** $[d]$. From another perspective, we observe that **FirstDiff** $[\log d]$ and **Greedy** $[d]$ result in a comparable maximum load, while the number of probes used by **FirstDiff** $[\log d]$ is exponentially smaller than that of **Greedy** $[d]$. In other words, we exhibit an algorithm that performs as well as an optimal algorithm, with significantly less computational requirements. We believe that our work has opened up a new family of algorithms that could prove to be quite useful in a variety of contexts spanning both theory and practice.

A number of questions arise out of our work. From a theoretical perspective, we are interested in developing a finer-grained analysis of the number of probes; experimental results suggest the number of probes used to place the i^{th} ball depends on the congruence class of i modulo n . From an applied perspective, we are interested in understanding how **FirstDiff** $[d]$ would play out in real world load balancing scenarios like cloud computing, where the environment (i.e. the servers, their interconnections, etc.) and the workload (jobs, applications, users, etc.) are likely to be a lot more heterogeneous and dynamic.

Acknowledgements

We are thankful to Anant Nag for useful discussions and developing a balls-in-bins library [7] that was helpful for our experiments. We are also grateful to Thomas Sauerwald for his helpful thoughts when he visited Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University. Finally, John Augustine and Amanda Redlich are thankful to ICERM for having hosted them as part of a semester long program.

References

- [1] Y. Azar, A. Z. Broder, A. R. Karlin, and E. Upfal, *Balanced allocations*, SIAM journal on computing, 29 (1999), pp. 180–200.
- [2] P. Berenbrink, A. Czumaj, A. Steger, and B. Vöcking, *Balanced allocations: The heavily loaded case*, SIAM Journal on Computing, 35 (2006), pp. 1350–1385.
- [3] P. Berenbrink, K. Khodamoradi, T. Sauerwald, and A. Stauffer, *Balls-into-bins with nearly optimal load distribution*, in Proceedings of the 25th ACM symposium on Parallelism in algorithms and architectures, ACM, 2013, pp. 326–335.
- [4] A. Czumaj and V. Stemann, *Randomized allocation processes*, Random Structures & Algorithms, 18 (2001), pp. 297–331.
- [5] S. Fu, C.-Z. Xu, and H. Shen, *Randomized load balancing strategies with churn resilience in peer-to-peer networks*, Journal of Network and Computer Applications, 34 (2011), pp. 252–261.
- [6] M. Mitzenmacher and E. Upfal, *Probability and computing: Randomized algorithms and probabilistic analysis*, Cambridge University Press, 2005.
- [7] A. Nag, *Problems in Balls and Bins Model*, Master’s thesis, Indian Institute of Technology Madras, India, 2014.

- [8] Y. Peres, K. Talwar, and U. Wieder, *The $(1 + \beta)$ -choice process and weighted balls-into-bins*, in Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms, Society for Industrial and Applied Mathematics, 2010, pp. 1613–1619.
- [9] M. Raab and A. Steger, *“Balls into Bins”–A Simple and Tight Analysis*, in Randomization and Approximation Techniques in Computer Science, Springer, 1998, pp. 159–170.
- [10] H. Shen and C.-Z. Xu, *Locality-aware and churn-resilient load-balancing algorithms in structured peer-to-peer networks*, Parallel and Distributed Systems, IEEE Transactions on, 18 (2007), pp. 849–862.
- [11] X.-J. Shen, L. Liu, Z.-J. Zha, P.-Y. Gu, Z.-Q. Jiang, J.-M. Chen, and J. Panneerselvam, *Achieving dynamic load balancing through mobile agents in small world p2p networks*, Computer Networks, 75 (2014), pp. 134–148.
- [12] K. Talwar and U. Wieder, *Balanced allocations: A simple proof for the heavily loaded case*, in Automata, Languages, and Programming, Springer, 2014, pp. 979–990.
- [13] B. Vöcking, *How asymmetry helps load balancing*, Journal of the ACM (JACM), 50 (2003), pp. 568–589.